JOURNAL OF APPROXIMATION THEORY 18, 346–349 (1976)

Simultaneous Approximation of a Function and Its Derivatives

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Communicated by E. W. Cheney

Received June 12, 1975

1. INTRODUCTION

Let $\mathbb{P}_0$ be the set of all polynomials of the form $\sum_{i=0}^{n} a_i x^i$ with $a_0 = c$ and $n \geqslant r + 1$, for some integer $r \geqslant 1$, where c is a nonzero constant. The polynomial $Q_0 \in \mathbb{P}_0$ is said to be a best approximation to zero, if

$$\max(\| Q_0 \|, \| D^r Q_0 \|) = \inf_{Q \in \mathbb{P}_0} \max(\| Q \|, \| D^r Q \|),$$

where $\| Q \| = (\int_0^t | Q(x)|^p \, dx)^{1/p}$, $p \geqslant 1$, and $D^r Q(x)$ is the rth derivative of $Q(x)$. It is natural to ask: Is $\| Q_0 \|$ equal to $\| D^r Q_0 \|$? In [2] it was shown that $\int_{-1}^{1} | Q_0(x)|^p \, dx = \int_{-1}^{1} | D^r Q_0(x)|^p \, dx$, for $p = 2$ and $c = 1$. In this note we will show that, in fact, $\| Q_0 \| = \| D^r Q_0 \|$, for $p \geqslant 1$.

2. GENERAL RESULTS

It is clear that we have the following trivial result.

LEMMA. *Let $\mathbb{P}_0$ be defined as above. If the polynomial $Q_0(x) \in \mathbb{P}_0$ is a best approximation to zero, then, for $p \geqslant 1$,*

$$\| Q_0 \| \geqslant \| D^r Q_0 \|. \tag{1}$$

Using a technique similar to that in [2], we can show that equality in (1) is actually attained.

THEOREM 1. *Let $\mathbb{P}_0$ be defined as above. If the polynomial $Q_0(x) \in \mathbb{P}_0$ is a best approximation to zero, then*

$$\| Q_0 \| = \| D^r Q_0 \| \qquad \text{for} \quad 1 \leqslant p < \infty.$$

Proof. Since $\| Q_0 \| \geqslant \| D^r Q_0 \|$, we need to show that $\| Q_0 \| > \| D^r Q_0 \|$ is impossible.

Suppose then that this inequality holds. Write $Q_0(x) = c - R(x)$, where $R(x) \in M$ and $M = \text{span}[x, \ldots, x^n]$. Then $R(x)$ is a best approximation to c from M with respect to $\| \cdot \|$ and hence we have

$$\int_0^1 x^k \frac{| Q_0(x)|^{p-1}}{\| Q_0 \|^{p-1}} \operatorname{sgn} Q_0(x)\, dx = 0 \qquad \text{for} \quad k = 1, \ldots, n. \tag{2}$$

Then $Q_0(x)$ must change sign at least n times. Since $Q_0(x)$ is a polynomial of degree $\leqslant n$, $Q_0(x)$ has n simple roots in $(0, 1]$, say $x_1, \ldots, x_n$. By Rolle's theorem, $DQ_0(x)$ vanishes at least once between any two zeros of $Q_0(x)$ and thus vanishes in at least $n - 1$ points. Continuing this argument and the fact that $D^r Q_0(x)$ is a polynomial of degree $\leqslant (n - r)$, we see that $D^r Q_0(x)$ has $(n - r)$ simple roots in $(0, 1)$, say $x_{r_1}, \ldots, x_{r_{n-r}}$. Obviously $x_1 < x_{r_1}, \ldots, x_{n-r} < x_{r_{n-r}}$. Moreover, we have

$$| c |/| a_n | = x_1 \cdot x_2 \cdot \cdots \cdot x_n$$

and

$$\left| \frac{r!\, a_r}{n(n-1) \cdots (n-r+1)\, a_n} \right| = x_{r_1} \cdots x_{r_{n-r}},$$

so

$$\begin{aligned} | r!\, a_r | &= n(n-1) \cdots (n-r+1)\, | a_n |\, x_{r_1} \cdots x_{r_{n-r}} \\ &= n(n-1) \cdots (n-r+1) \frac{x_{r_1} \cdots x_{r_{n-r}} | c |}{x_1 \cdots x_n} \\ &> n(n-1) \cdots (n-r+1) \left(\frac{| c |}{x_{n-r+1} \cdots x_n} \right). \end{aligned}$$

Hence

$$\begin{aligned} (1/| c |)\, | D^r Q_0(0)| &= r!\, | a_r |/| c | \\ &> n(n-1) \cdots (n-r+1) \left(\frac{1}{x_{n-r+1} \cdots x_n} \right) > 1. \end{aligned}$$

From (2), we have

$$\int_0^1 (D^r Q_0(0)\, Q_0(x) - c D^r Q_0(x)) \frac{| Q_0(x)|^{p-1}}{\| Q_0 \|^{p-1}} \operatorname{sgn} Q_0(x)\, dx = 0. \tag{3}$$

Then, using the Hölder's inequality and (3), we obtain

$$\begin{aligned} \left| \frac{1}{c} \right| | D^r Q_0(0)|\, \| Q_0 \| &= \left| \int_0^1 D^r Q_0(x) \frac{| Q_0(x)|^{p-1}}{\| Q_0 \|^{p-1}} \operatorname{sgn} Q_0(x)\, dx \right| \\ &\leqslant \| D^r Q_0 \| \left(\int_0^1 \frac{| Q_0(x)|^{(p-1)q}}{\| Q_0 \|^{(p-1)q}}\, dx \right)^{1/q} = \| D^r Q_0 \|. \end{aligned}$$

As $|D^r Q_0(0)|/|c| > 1$, we would have

$$\| D^r Q_0 \| > \| Q_0 \|.$$

This contradicts the assumption, which completes the proof of the theorem.

THEOREM 2. *The polynomial $Q_0(x)$ of Theorem 1 does not have zeros of multiplicity r in $(0, 1)$, for $1 < p < \infty$.*

Proof. Suppose there exists $x_0 \in (0, 1)$ such that $D^i Q_0(x_0) = 0$ for $i = 0, 1, \ldots, r - 1$. Then, using the Hölder's inequality, we obtain

$$\begin{aligned}
\int_0^1 | Q_0(x)|^p \, dx &= \int_0^1 \left| \int_{x_0}^x DQ_0(t)\, dt \right|^p dx \\
&\leqslant \int_0^1 \left(\left(\int_{x_0}^x | DQ_0(t)|^p \, dt \right)^{1/p} \cdot | x - x_0 |^{1/q} \right)^p dx \\
&< \int_0^1 | x - x_0 |^{p/q} \, dx \cdot \int_0^1 | DQ_0(x)|^p \, dx \\
&\leqslant (1/p) \int_0^1 | DQ_0(x)|^p \, dx.
\end{aligned}$$

Similarly, we have

$$\int_0^1 | D^{i-1} Q_0(x)|^p \, dx < (1/p) \int_0^1 | D^i Q_0(x)|^p \, dx \qquad \text{for} \quad i = 2, 3, \ldots, r.$$

Therefore

$$\int_0^1 | Q_0(x)|^p \, dx < (1/p^r) \int_0^1 | D^r Q_0(x)|^p \, dx.$$

Since $Q_0(x)$ is a best approximation, this contradicts the equality of Theorem 1, which proves the theorem.

Finally, by virtue of Theorem 1 and one of the results in [1], we have the following Characterization Theorem.

THEOREM 3. *The polynomial $Q_0(x) \in \mathbb{P}_0$ is a best approximation to zero if and only if there exist positive numbers λ, μ such that $\lambda + \mu = 1$,*

$$\int_0^1 x^k \frac{| Q_0(x)|^{p-1}}{\| Q_0 \|^{p-1}} \operatorname{sgn} Q_0(x) \, dx = 0 \qquad \textit{for} \quad k = 1, 2, \ldots, r - 1,$$

and

$$\begin{aligned}
\lambda \int_0^1 x^k \frac{| Q_0(x)|^{p-1}}{\| Q_0 \|^{p-1}} \operatorname{sgn} Q_0(x) \, dx + \mu k \times \cdots \times (k - r + 1) \\
\times \int_0^1 x^{k-r} \frac{| D^r Q_0(x)|^{p-1}}{\| D^r Q_0 \|^{p-1}} \operatorname{sgn} D^r Q_0(x) \, dx = 0 \qquad \textit{for} \quad k = r, r + 1, \ldots, n.
\end{aligned}$$

ACKNOWLEDGMENT

I am indebted to the referee for his valuable suggestions.

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